

Study of the disordered one-dimensional contact process.

Raffaele Caferio^{1,3}, Andrea Gabrielli^{2,3}, and Miguel A. Muñoz³

¹ *Max-Planck Institut für Physik komplexer Systeme, Nöthnitzer strasse 38, D-01187 Dresden, Germany*

² *Dipartimento di Fisica Università degli Studi "Tor Vergata", v.le della Ricerca Scientifica 1, 00133 Roma Italy*

³ *Dipartimento di Fisica e unità INFN, Università di Roma "La Sapienza", P.le A. Moro 2, I-00185 Roma, Italy*
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Abstract

New theoretical and numerical analysis of the one-dimensional contact process with quenched disorder are presented. We derive new scaling relations, different from their counterparts in the pure model, which are valid not only at the critical point but also away from it due to the presence of generic scale invariance. All the proposed scaling laws are verified in numerical simulations. In addition we map the disordered contact process into a Non-Markovian contact process by using the so called Run Time Statistic, and write down the associated field theory. This turns out to be in the same universality class as one derived by Janssen for the quenched system with a Gaussian distribution of impurities. Our findings here support the lack of universality suggested by the field theoretical analysis: generic power-law behaviors are obtained, evidence is shown of the absence of a characteristic time away from the critical point, and the absence of universality is put forward. The intermediate sublinear regime predicted by Bramson et al. is also found.

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I. INTRODUCTION

As first conjectured by Janssen and Grassberger [1], many numerical and analytical studies have established clearly that all the systems exhibiting a continuous transition into an *unique* absorbing state, without any other extra symmetry or conservation law, belong into the same universality class, namely, that of the contact process (CP) [2,3]. That conjecture has been extended to include multicomponent systems [4], and also systems with an infinite number of absorbing states [5]. Among many other models in this broad class are the following: directed percolation [2,3], the contact process [6], catalytic reactions on surfaces [7], the spreading of epidemics, and branching annihilating random walks [8]. The Reggeon Field Theory (RFT) is the minimal continuous theory capturing the key features of this universality class [9,1], (which is often referred to as directed percolation (DP hereafter) universality class).

Despite of its theoretical importance, no experiment has succeeded so far in identifying critical exponents compatible with the predicted DP values. This could be due to the fact that real systems are never pure, i.e, they present impurities, dilution or other forms of disorder. The question arises of how disorder affects the critical behavior of DP-like systems. That problem was first posed by Kinzel [10] and studied numerically by Noest [11,12] who showed using a Harris criterium [13] that quenched disorder changes the critical behavior of DP systems in spatial dimensions below $d = 4$. He also demonstrated that in $d = 1$ generic power law (generic scale invariance) can be observed, and that in $d = 2$ a Griffiths-like phase [14] can appear when the impurities take the form of dilution [11]. This same problem has been recently tackled by Dickman and Moreira in an interesting series of papers [15,16] where they have pointed out the presence of logarithmic time-dependences in $d = 2$ and a possible violation of scaling.

In any case, the dynamics in impure DP-systems is well established to be extremely slow: due to the presence of impurities, a system that globally is in the absorbing phase, can include regions that take locally parameter values that correspond to the active regime in the analogous pure system. The presence of these zones makes it difficult for the system to relax to the absorbing state, and consequently it decays in a slow fashion: i.e, exhibiting power laws in $d = 1$ [11,12], and logarithmically in $d = 2$ [15,16], but not exponentially as generically occurs in pure systems away from the critical point.

The problem of temporally disordered systems with absorbing states has also been recently investigated, with apparently striking conclusions [17].

At a theoretical level a field-theory analysis for this class of impure systems has recently been derived by Janssen [18]. His works corrects a previous incomplete analysis [19], and concludes from an epsilon expansion around the upper critical dimension, $d = 4$, that the renormalization group flow equations exhibit only runaway trajectories, and therefore there is no stable (perturbative) fixed point (nothing can be concluded about non-perturbative fixed points). This can be seen as an evidence that no universal critical behavior is expected in this class of models.

In this paper we revisit the impure one-dimensional problem, and look at it within a new perspective. In particular, we analyze the presence or absence of scaling laws in analogy with the two-dimensional results recently presented by Dickman and Moreira, study the universality of critical exponents and the scaling relations they obey, and verify the presence

of a sublinear regime predicted by Bramson et al. [20]. On the other hand, we present a non-Markovian representation of this class of systems that shows the same phenomenology, and derive from it a field theory that turns out to be equivalent to the one derived by Janssen. From the field theory we obtain new relations among exponents.

II. THE MODEL

In the standard contact process [6,2] each site of a d -dimensional lattice is either 'occupied' or 'vacant'. In its discrete-time version, an occupied site is extracted randomly at each time step; it generates an offspring with probability p , or disappears with complementary probability $1 - p$. The offspring occupies a randomly chosen nearest neighbor: if it was empty it becomes occupied, while the system remains unchanged if the neighbor was already occupied. In the disordered contact process the probability p changes from site to site, is fixed in time, and is extracted from a distribution $\Pi(p)$. Through this paper we consider in particular:

$$\Pi(p, a) = ap^{a-1}, \quad (1)$$

for which

$$\langle p \rangle = \frac{a}{a+1}, \quad (2)$$

in this way a acts as a control parameter. For large values of a , the creation probability is large, and the system is in the active phase. Contrarily, for sufficiently small values of p the system decays into the absorbing state. We have chosen the previous distribution for technical reasons: it simplifies the application of the Run Time Statistic [21] that we employ to study the model.

The central magnitudes, usually considered in this kind of systems are of two types: magnitudes measured in analysis with homogeneous initial conditions, and those measured studying the spreading of a localized 'seed' into the otherwise empty space [22].

In the first group, we determine the stationary order parameter (defined as the average density of particles in the stationary state), n , the correlation time, τ , and the correlation length, ξ .

In the second group we study: the total number of occupied sites in the lattice (averaged over all the runs including those which have reached the absorbing state) as a function of time, $N(t)$, the overall surviving probability $P_s(t)$, that is, the probability that the system has not reached the absorbing state at time t , and the mean square distance of spreading from the origin of the surviving trials as a function of time, $R^2(t)$ [22].

Right at the critical point of pure systems, we have:

$$N(t) \propto t^\eta, \quad P_s(t) \propto t^{-\delta}, \quad R^2(t) \propto t^z, \quad \text{and} \quad n(t) \propto t^{-\theta}, \quad (3)$$

and at a small distance Δ from the critical point,

$$n \propto \Delta^\beta, \quad \tau \propto \Delta^{-\nu_t}, \quad \xi \propto \Delta^{-\nu_x} \quad (4)$$

which define the set of critical exponents we are interested in. In pure systems the following scaling relations hold:

$$\eta + \delta + \theta = d z/2, \quad \delta = \theta, \quad z = 2\nu_x/\nu_t, \quad \text{and} \quad \theta = \beta/\nu_t \quad (5)$$

some of these expressions have to be modified for the disordered model as we will show (see [3,1] and [25] and references therein).

III. NON-MARKOVIAN REPRESENTATION

We start our analysis of the model by mapping it into a non-Markovian model. The idea of representing a model with quenched disorder by means of an effective non-Markovian equation, i.e., with memory, including no disorder, is not a new one. A complete theory for doing so has been developed in [21]: it has been named the Run Time Statistic (RTS), and has proven to be an useful tool in the study of fractals with quenched disorder [23], and self-organized models with extremal dynamics [24].

The central idea of the RTS can be exemplified by its application to the random random walker (RRW) [26]. The RRW is defined in the following way: a standard one-dimensional random walker is considered, with the only difference that the probabilities of jumping to the right, q , or to the left, $1 - q$, change from site to site, are quenched, and extracted from a certain probability distribution $P(q)$. The probability that at a given site, characterized by a given value of q , visited n times by the walker, the walker has jumped k times to the left is given by the binomial distribution:

$$P(k|q, n) = \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}. \quad (6)$$

Using the Bayes inversion formula for the inversion of conditional probabilities one can calculate the probability that at a given site the probability q takes a particular value between q and $q + dq$ from the knowledge of k after n jumps [27,26]

$$P(q + dq|n, k) = \frac{(n+1)!}{k!(n-k)!} q^k (1-q)^{n-k} P(q) dq. \quad (7)$$

An effective transition probability can be accordingly defined as:

$$q(n, k) = \int dq q P(q|n, k). \quad (8)$$

This equation gives the effective probability for the walker to jump to the right in its $n + 1$ visit to a given site, conditioned to the fact that in n previous visits it jumped k times to the right.

Observe that the distribution eq.(7) changes with time (with n); the information about the history of the system is contained in the effective transition probabilities (that change from site to site). This is usually called *run time statistic* [21].

Let us now apply the previously described method to the disordered contact process. At each site the value of p (that plays now a role analogous to q in the RRW), is extracted from the distribution eq. (1) [28]; it is straightforward to verify that

$$P(p + dp|n, k) = p^{k+a-1}(1-p)^{n-k} \frac{(n+1)!}{(k+a-1)!(n-k)!} dp \quad (9)$$

where now: n is the number of times that a given site has been chosen to try an evolution step, and k is the total number of times in which an offspring has been generated (obviously $1-k$ is the number of events in which the site has become empty). Therefore the effective parameter p at the site under consideration is:

$$\langle p \rangle = \int_0^1 dp \, p \, P(p + dp|n, k) = \frac{k+a}{n+a+1}. \quad (10)$$

Note that the distribution of effective values of the probability $\langle p \rangle$ for any arbitrary n as large as wanted does not collapse to a delta function, but converges asymptotically to the distribution (1) [26].

IV. FIELD THEORY

Using the previously derived non-Markovian approach we can easily derive an associate field theory. Let us first consider the standard Reggeon field theory describing the universality class of the pure contact process [1,9].

$$S[\phi, \psi] = \int dr^d \int_0^\infty dt \left[\lambda \psi(x, t)^2 \phi(x, t) - \psi(x, t) \left(\partial_t \phi - \mu^2 \phi - \lambda \phi(x, t)^2 - \nabla^2 \phi(x, t) \right) \right]. \quad (11)$$

The coefficient of the linear term, μ^2 (the *mass* in a field theoretical language), depends linearly on the creation probability p . A large p renders the contact process supercritical, and so does a value of μ^2 above its critical value. Observe that at any time the renormalized value of μ^2 at a given point x is given by the expectation value of $\psi(x, t)\phi(x, t)$. In order to implement the dependence of p on the history at each point x , we can perform the following substitution:

$$\mu^2 \rightarrow \mu_{mod}^2(x, t) = \mu^2 + \gamma \int_0^t d\tau \psi(x, \tau) \phi(x, \tau), \quad (12)$$

that is, at every time step, the *modified* value of the linear coefficient, μ_{mod}^2 , is given by its original value corrected by a time dependent term: the expectation value of $\psi(x, t)\phi(x, t)$ over the previous history of the system. Observe that γ acts as a normalization factor. In this way the action becomes:

$$S_M[\phi, \psi] = \int dr^d \int_0^\infty dt \left[\lambda \psi^2 \phi - \psi (\partial_t \phi - \mu^2 \phi - \lambda \phi^2 - \nabla^2 \phi) + \gamma \psi \phi \int_0^t d\tau \psi(x, \tau) \phi(x, \tau) \right] \quad (13)$$

where the dependence of the fields on x and t has been omitted for the economy of notation.

On the other hand considering the standard Reggeon Field theory, eq. (11), with a site-dependent quenched mass coefficient, $\mu^2(x)$, and a Gaussian distribution of 'masses' with mean μ^2 , and variance $\langle \mu^2(x) \mu^2(x') \rangle = 1/2f \delta(x - x')$, one gets after averaging over the disorder (for what one just has to perform a Gaussian integral [18])

$$S_d[\phi, \psi] = \int dr^d \left[\int_0^\infty dt \left[\lambda \psi^2 \phi - \psi(\partial_t \phi - \mu^2 \phi - \lambda \phi^2 - \nabla^2 \phi) \right] \right] + f \left[\int_0^\infty dt \psi(x, t) \phi(x, t) \right]^2. \quad (14)$$

Observe that eq. (14) coincides with eq. (13) except for a time integration limit and the value of the coefficients. This difference states that the non-Markovian approach reproduces the exact result in the large time limit. Such a difference between the two field theories can be argued to be irrelevant.

Naive power counting arguments show that all the three non-linearities in eq. (13) can be renormalized in $d = 4$. This result is consistent with the Harris criterium [13] presented by Kinzel and Noest in [10,11], which states that quenched spatial disorder affects the critical behavior of the contact process and models in the same universality class below $d = 4$.

The detailed renormalization procedure of eq. (14) can be found in [18], where it is concluded that no stable fixed point exists below $d = 4$. This result is found performing an epsilon expansion around the critical dimension, and is, therefore, valid only in a perturbative sense. The implication of this fact is that either there is a strong coupling fixed point, or there is no fixed point at all. This latter possibility could reveal the presence of discontinuous transitions (that have never been observed numerically) or, in any case, a lack of universality. More rigorous conclusions are not available at this point from the field theoretical analysis.

V. SCALING LAWS

The field theory we have written down can also be used as a starting point to derive scaling relations. From eq. (11) (or using other standard scaling arguments), it is easy to derive that in the active regime:

$$\eta + \delta + \theta = dz/2. \quad (15)$$

Let us derive it analogous for the impure (non-Markovian) model here using simple arguments: as $N(t)$ is obtained averaging over all the runs, it can be written as $N(t) = N_s(t)P_s(t) + 0 * (1 - P_s(t))$ where $N_s(t)$ is the total number of particles calculated averaging only over surviving runs. Consequently, one gets, $N_s(t) \approx t^{\eta+\delta}$. After creating a perturbation, if a growing cluster of occupied sites is generated, the radius of such a cluster grows as $R \propto t^{z/2}$ and its volume as $R^d \propto t^{dz/2}$. From the two previous expressions, the density of particles inside the cluster goes like $t^{\eta+\delta-dz/2}$. But this density inside the cluster scales as $t^{-\theta}$ by definition of θ , therefore we have obtained eq. (15).

The previous expression is valid only at the critical point in the pure model, where scale invariance is expected. Contrarily in the impure model, where generic scale invariance is expected, the previous argument is valid in all the active phase, in which growing clusters are typically generated from localized seeds.

On the other hand, in the absorbing phase, typically initial seeds are located in locally absorbing regions and die out exponentially. However, there is a probability for the initial seed of 'landing' in a locally active cluster. When the perturbation gets out of these clusters dies out exponentially fast. But inside these finite clusters the local stationary density is reached in a finite time. Therefore we can substitute formally θ by zero in eq. (15) and get:

$$\eta + \delta = dz/2 \quad (16)$$

(note that this does not mean that θ is zero).

On the other hand using the symmetry of the Lagrangean under the exchange of the fields ϕ and ψ it is not difficult to get

$$\delta = \theta \quad (17)$$

as in the pure model (see [25] for a review of the underlying ideas). Observe that the previous symmetry, present in the Reggeon field theory, is not broken by the introduction of the non-Markovian term, i.e. by the quenched impurities. Therefore:

$$\eta + 2\delta = dz/2 \quad (18)$$

in the active regime of our model, as well as in the critical point of the pure model.

In the active phase, starting from an homogeneous distribution the system relaxes to its stationary state also as a power law. By definition of the active regime the surviving probability does not go to zero for large times, and as $P_s(t)$ is a monotonously decreasing function of time, we get that $\delta = 0$ all along the active phase. Using the scaling relation eq. (17) we also get $\theta = 0$. This simplifies eq. (18) to

$$\eta = dz/2. \quad (19)$$

For completeness let us point out that the exponent \hat{d} calculated in [11] is easily found to be related to the exponents we have defined by

$$\hat{d} = 1 + \eta + \delta. \quad (20)$$

For that it is enough to observe that \hat{d} is the exponent of a time integral of the total number of particles averaged over the surviving runs.

Summing up the main conclusions of this section are:

- In the active phase: $\eta = dz/2$ and $\delta = \theta = 0$.
- In the absorbing phase: $\eta + \delta = dz/2$, and $\delta = \theta \neq 0$.

VI. MONTE CARLO RESULTS

We have performed extensive Monte Carlo simulations of the contact process with quenched impurities distributed according to eq. (1), as well as of the associated non-Markovian contact process defined by eqs. (9) and (10). Spreading experiments have been performed in lattices large enough so the occupied region does not reach the system limits. Experiments started with a random homogeneous initial condition have been performed in system sizes up to $L = 10^4$ with periodic boundary conditions. Most of the results presented in what follows correspond to $L = 10^3$. At every time step a particle is randomly chosen and the dynamics proceeds in the way explained in the model definition section; after each step the time variable t is increased in $1/N(t)$; i.e. when all the particles are updated once on average, the time increases in one unit. Simulations are run long enough as to let the

system relax to its stationary state in the active phase ($t \approx 1.6 \times 10^5$ time steps). The different magnitudes are obtained by averaging over many independent runs (from 10^2 for large values of $\langle p \rangle$, where most realizations die at late times and we can easily collect a good statistics, to 10^5 for small values of $\langle p \rangle$, where many realizations die at early times). All the forthcoming discussions are valid for both the model with quenched disorder and the non-Markovian equivalent model: the results are identical within the numerical accuracy. In fact, as it is shown in Fig. 1, the long time distribution of values of $\langle p \rangle$ in the non-Markovian model is verified to converge to the distribution in the quenched model, eq. (1). To avoid repetition we discuss both cases as a whole, and present figures for both the disordered and the non-Markovian model.

The main results we have obtained are the following:

A. Homogeneous initial conditions

The density of particles $n(t)$ ($n(t) = N(t)/L$) decays in time as shown in Fig.2. Observe that for large enough values of $\langle p \rangle$ the curves converge to a stationary value, that is, their derivative with respect to time converges to zero asymptotically. On the other hand for small values of $\langle p \rangle$ the curves decay like power laws with non-universal exponents that depends on $\langle p \rangle$; $n(t) \propto t^{-\theta(\langle p \rangle)}$. In Fig.3 we show the asymptotic exponent $\theta(\langle p \rangle)$ as a function of $\langle p \rangle$, for both the disordered and the non-Markovian model. Observe that it decays continuously from its maximum value in the absorbing state to a very small value (compatible with zero), and it is zero in the active regime. Observe also the difficulty to locate accurately the critical point. Usually, i.e. in pure systems, $n(t)$ decays exponentially in the absorbing phase, converges to a constant in the active phase; and decays as a power law only at the critical point. Consequently there is a neat criterium to identify the critical point: power laws are the hall-mark of criticality. In the impure model, instead, the generic presence of power laws makes the determination of the critical point a more delicate issue, but at the same time a more irrelevant one.

Two possible scenarios are compatible with the data we have obtained: in the first one $\theta(\langle p \rangle)$ is a continuous function of $\langle p \rangle$ and the point in which it 'touches' zero for the first time corresponds to the critical point. The second possibility is that there is a discontinuous jump at the critical point; i.e. the curves in Fig. 3 would not be continuous; that would imply a non-zero value of θ at the transition point. Even though from our numerics it is not possible to resolve the previous dilemma, we are tempted to conclude that the first possibility is the right one, based on the small values of $\theta(\langle p \rangle)$ we measure in the vicinity of the critical point, and to the fact that the slopes are always observed to change smoothly with $\langle p \rangle$, therefore, no 'jump' is expected to occur. In any case, from the numerics θ can be expressed at the critical point as $\theta(\langle p \rangle_c) = 0.02 \pm 0.05$, with $\langle p \rangle_c = 0.71 \pm 0.01$ (see below).

In Fig.4 we plot the asymptotic density n as a function of $\langle p \rangle$, together with a power law fit. The best fit is obtained taking $\langle p \rangle_c = 0.705$ for the critical effective parameter, and gives $\beta = 0.29 \pm 0.01$ for both the disordered and the non-Markovian model. In Fig. 5 we check the consistence of our assumption on $\langle p \rangle_c$, by representing in a log-log plot $n^{1/\beta}$ as a function of $\Delta = \langle p \rangle - \langle p \rangle_c$, with $\beta = 0.29$. The extrapolation to zero of $n^{1/\beta}$ gives $\langle p \rangle_c = 0.71 \pm 0.01$, consistent with our previous assumption. Observe that the value

of β we find is very different from the one obtained by Noest for a different distribution of impurities, $\beta = 1.75 \pm 0.1$ [11]. We interpret this fact as a consequence of the absence of universality predicted by the field theory analysis.

From the previous analysis (which agrees perfectly with the theoretical predictions) we can extract the following striking conclusion: as β assumes a finite value and θ is compatible with zero, using the scaling relation $\theta = \beta/\nu_t$ we get that either ν_t is infinity or takes an extremely large value. Observe that Noest measured $\nu_t = 4.0 \pm 0.5$ [11] which is an atypically large value. In fact, an analogous result has been obtained in the two-dimensional version of the model [16]; Dickman and Moreira showed that as a matter of fact the exponent ν_t is not even defined. This is a straightforward consequence of the fact that the correlation functions do not decay exponentially in the absorbing phase, but as power laws, i.e., there is no associated characteristic time, and therefore ν_t is undefined (or formally $\nu_t = \infty$).

In order to further explore this issue we have measured the two-time correlation functions, $\langle n(t_0)n(t_0+t) \rangle - \langle n(t=\infty) \rangle^2$, for large times and different values of $\langle p \rangle$ for the disordered model. The results are presented in Fig. 6. First we observe that in all the cases, i.e. above, below and at the critical point, we get power law behaviors, therefore there is no characteristic time scale. Second, for a fixed value of $\langle p \rangle$ and varying t_0 we observe different transient regimes but the asymptotic behavior does not depend on t_0 for large enough times. This is a prove that the model does not exhibits aging [29]; therefore even though the field theory representing the model is non-Markovian (i.e., the two-time correlation functions cannot be expressed only as a function of the time difference), the system relaxes to an aging-free state.

This analysis can be interpreted as further supporting the guess that $\theta = 0$ at the critical point. Otherwise, using the scaling relations, we would get a finite ν_t and consequently an exponential decay of the two-time correlation function.

B. Spreading

In Fig. 7,8 and 9 we present the evolution of the magnitudes defined in eq. (3) for the spreading experiments, while in tables 1 and 2 we give a summary of the values of all the scaling exponents for the magnitudes we have studied, including the exponent θ , related to homogeneous initial conditions, together with a checking of the scaling relations between the exponents.

Observe that the three magnitudes $N(t)$, $P_s(t)$ and $R^2(t)$ present generic power law decays. In table 1 we show the values of the associated exponents η , δ and z for different values of $\langle p \rangle$. Note that all the scaling laws predicted in the previous section are satisfied generically within the accuracy limits. In particular, observe that right at the critical point and in the active phase we get a value of δ compatible with $\delta = 0$, and therefore satisfying the predicted scaling relation: $\theta = \delta = 0$. As a byproduct we obtain a confirmation of a result obtained some time ago by Bramson et al. [20]. They demonstrated that an impure version of the one-dimensional contact process exhibits an intermediate phase, i.e., a region in the active phase in which R^2 grows slower than t^2 . This is accordingly called the *sublinear regime* [30]. We observe sublinear growth in all the active phase: only in the limit $\langle p \rangle = 1$ ($a \rightarrow \infty$) we get linear growth: the intermediate phase covers the whole active phase. Therefore the presence of such a sublinear regime seems to be a generic feature of

impure one-dimensional systems with absorbing states. Our results could be compared with those obtained by Noest for a different impurity distribution. He got, at the critical point, $\hat{d} = 1.28 \pm 0.03$, which using eq. (20), implies $\eta + \delta = 0.28 \pm 0.03$, to be compared with the value 0.25 ± 0.02 that we measure (Tabb. I, II). On the other hand for z Noest measured $z = 1.44 \pm 0.06$ (using the relation $z = 2\nu_x/\nu_t$), and we get $z = 0.58 \pm 0.02$, indicating again a high degree of non-universality (Tabb. I, II).

As a last observation we want to point out that the curves for $\langle R^2(t) \rangle$ in the absorbing phase (see Fig. 9) do not seem to have reached their stationary value in the time scale under consideration. Thus, the values of z we give in the tables must be taken carefully, since the error bars on z are quite large. In fact, our results seem compatible with $z = 0$, asymptotically. Observe also, that the combination $\eta + \delta$ in the absorbing phase gives a small exponent that could also be compatible with zero asymptotically.

In any case, all the predicted scaling relations among exponents are perfectly satisfied at, above and below the critical point.

VII. CONCLUSIONS AND SUMMARY

We have studied under different perspectives the disordered contact process. First we have mapped it into a pure model with memory, that reproduces all the phenomenology of the original model. From this new non-Markovian model we write down a simple field theory, that is in the same universality class as one presented previously for the model with Gaussian-distributed quenched disorder. Using the field theory we have derived a set of scaling relations not only at the critical point but also in the active and absorbing phases where scale invariance is also observed. Our theoretical predictions are confirmed in extensive Monte carlo simulations: in particular we have shown the equivalence of the disordered and the non-Markovian model, the generic presence of scale invariance, the existence of a sublinear-growth regime, verified the absence of a characteristic time scale, and verified all the predicted scaling laws.

In a future work we plan to further exploit the mapping into the non-Markovian model to obtain some other analytical results. In particular, we pretend to apply real-space renormalization methods to the one-dimensional model, and try to understand the generic scale invariance from a renormalization perspective.

VIII. ACKNOWLEDGEMENTS

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TABLES

p	η	δ	z	θ	$\eta + \delta - dz/2$	$\eta - dz/2$	$\delta - \theta$
0.5	-0.52 ± 0.02	0.61 ± 0.02	0.12 ± 0.07	0.57 ± 0.02	0.03 ± 0.07	—	0.04 ± 0.04
0.55	-0.32 ± 0.02	0.48 ± 0.02	0.14 ± 0.07	0.44 ± 0.02	0.09 ± 0.07	—	0.04 ± 0.04
0.66	-0.03 ± 0.01	0.10 ± 0.01	0.16 ± 0.06	0.13 ± 0.02	-0.01 ± 0.04	—	-0.03 ± 0.03
0.70	0.19 ± 0.02	0.05 ± 0.01	0.57 ± 0.02	0.02 ± 0.05	-0.04 ± 0.04	—	0.03 ± 0.06
0.71	0.25 ± 0.02	0.0 ± 0.01	0.58 ± 0.02	0.02 ± 0.05	—	-0.04 ± 0.03	-0.02 ± 0.06
0.725	0.35 ± 0.02	0	0.72 ± 0.02	0	—	-0.01 ± 0.03	0
0.75	0.53 ± 0.02	0	1.10 ± 0.01	0	—	-0.02 ± 0.03	0
0.8	0.92 ± 0.02	0	1.79 ± 0.01	0	—	0.02 ± 0.03	0
0.85	0.99 ± 0.02	0	1.99 ± 0.01	0	—	-0.01 ± 0.03	0
0.95	1.00 ± 0.02	0	2.00 ± 0.01	0	—	0.0 ± 0.03	0

TABLE I. Values of the scaling exponents for different values of p (disordered model).

p	η	δ	z	θ	$\eta + \delta - dz/2$	$\eta - dz/2$	$\delta - \theta$
0.5	-0.50 ± 0.02	0.54 ± 0.02	0.10 ± 0.08	0.57 ± 0.02	-0.01 ± 0.08	—	-0.03 ± 0.04
0.55	-0.37 ± 0.02	0.43 ± 0.02	0.12 ± 0.08	0.43 ± 0.02	0.0 ± 0.08	—	0.0 ± 0.04
0.66	-0.08 ± 0.02	0.10 ± 0.01	0.14 ± 0.07	0.11 ± 0.02	-0.05 ± 0.06	—	-0.01 ± 0.03
0.70	0.19 ± 0.02	0.03 ± 0.01	0.57 ± 0.02	0.02 ± 0.04	-0.06 ± 0.04	—	0.01 ± 0.05
0.71	0.24 ± 0.02	0.0 ± 0.01	0.58 ± 0.02	0.01 ± 0.04	—	-0.05 ± 0.03	-0.01 ± 0.05
0.725	0.34 ± 0.02	0	0.81 ± 0.04	0	—	-0.06 ± 0.04	0
0.75	0.47 ± 0.02	0	1.03 ± 0.02	0	—	-0.04 ± 0.03	0
0.8	0.93 ± 0.02	0	1.79 ± 0.01	0	—	0.04 ± 0.02	0
0.85	0.99 ± 0.02	0	1.96 ± 0.01	0	—	0.01 ± 0.02	0
0.95	1.00 ± 0.02	0	2.00 ± 0.01	0	—	0.0 ± 0.02	0

TABLE II. Values of the scaling exponents for different values of p (non-Markovian model).

FIGURES

FIG. 1. Distribution of $\langle p \rangle$ for large times in the non-Markovian model $\Pi_M(p)$ compared with the fixed distribution of the disordered model $\Pi_Q(p)$ for $a = 2$.

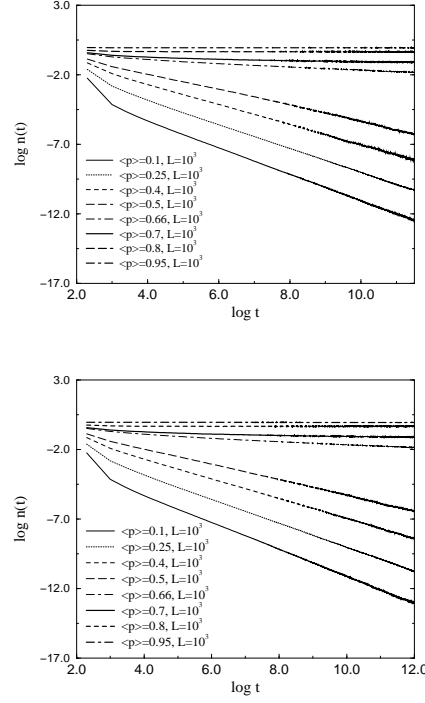
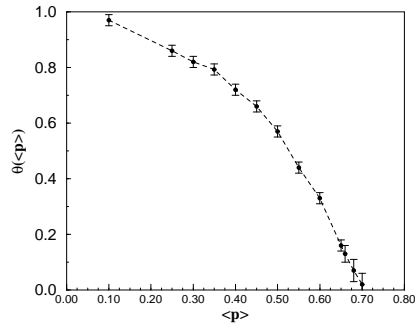


FIG. 2. Decay of the density of occupied sites for different values of $\langle p \rangle$ as a function of time for the disordered model (upper plot), and for the non-Markovian model (lower plot).



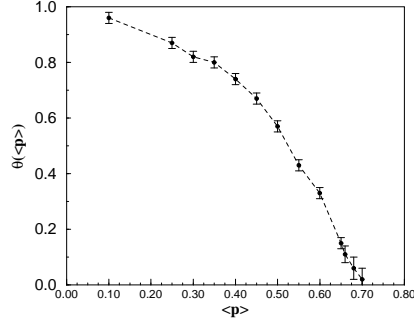


FIG. 3. Value of the exponent $\theta(< p >)$ as a function of $< p >$ for the disordered model (upper plot), and for the non-Markovian model (lower plot).

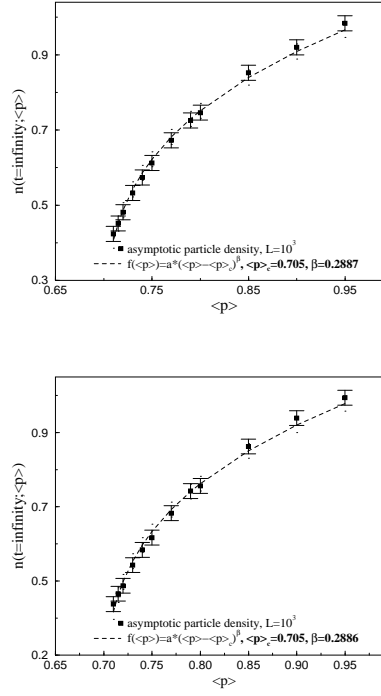


FIG. 4. Stationary value of the density as a function of $< p >$ for the disordered model (upper plot) and for the non-Markovian model (lower plot). A power law fit for the scaling of $n(t = \infty; < p >)$ is shown in the figure.

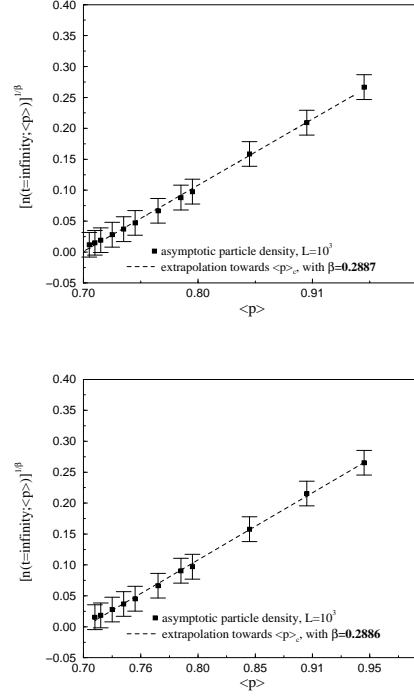
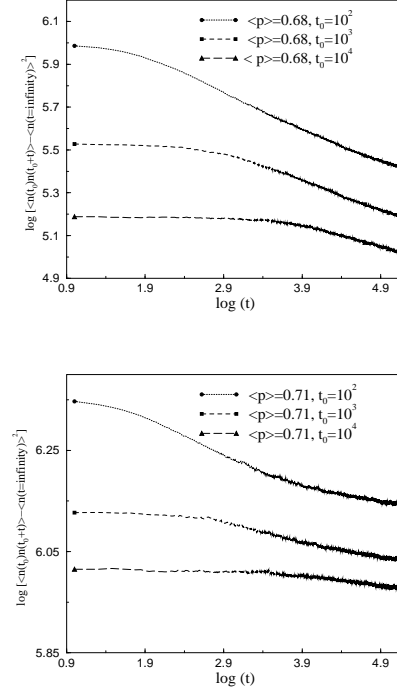


FIG. 5. Log-log plot of $n(t = \infty; \langle p \rangle)^{1/\beta}$ as a function of $\langle p \rangle$ for the disordered model (upper plot) and for the non-Markovian model (lower plot). The value of β is that given by the fit in Fig. 4. The extrapolation to $n(t = \infty) = 0$ gives $p_c = 0.71 \pm 0.01$.



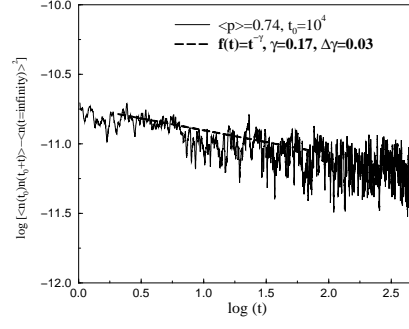


FIG. 6. Decay of the two-time density correlation function for different initial times (disordered model) as a function of the time difference t , and different values of $\langle p \rangle$: $\langle p \rangle = 0.68$ in the absorbing phase (upper figure), $\langle p \rangle = 0.71$ at critical point (central figure), and $\langle p \rangle = 0.74$ in the active phase (lower figure).

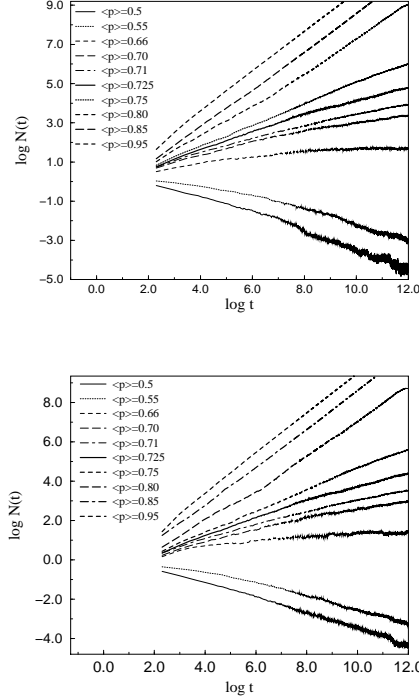


FIG. 7. Averaged total number of particles for spreading experiments, with different values of $\langle p \rangle$, as a function of time for the disordered model (upper plot), and for the non-Markovian model (lower plot).

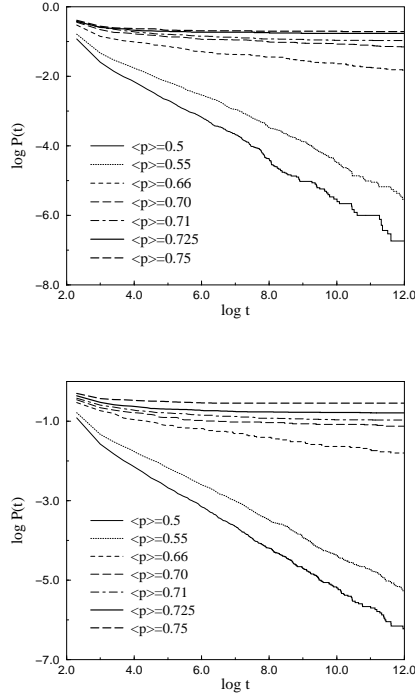


FIG. 8. Surviving probability for spreading experiments, with different values of $\langle p \rangle$, as a function of time for the disordered model (upper plot), and for the non-Markovian model (lower plot).

FIG. 9. Averaged square distance from the initial seed, with different values of $\langle p \rangle$, in a spreading experiment as a function of time for the disordered model (upper plot), and for the non-Markovian model (lower plot).

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[30] In pure systems, an initial seed spreads asymptotically in the active zone with a law $R^2 \propto t^2$. In $d = 2$ strong evidence has been found that no sublinear regime exists [16].

